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NUMERICAL INVESTIGATION OF CONVECTIVE FLOWS OF A HIGHLY INHOMOGENEOUS GASEOUS MEDIUM

I. M. Kozlov, A. E. Suvorov, and V. I. Tyukaev

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Equations of convection in a gas are analyzed and a computational algorithm that is efficient when the condition $\mu gh/RT \ll 1$ is satisfied for an arbitrary (fairly large) temperature drop is proposed. Calculations of convection occurring in steady nonuniform heating of walls are performed for axisymmetric argon-containing volumes of different shape. Results of these calculations demonstrate the establishment of steady and periodic motions of the gas.

Introduction. A standard approximation for investigating convection is the Boissinesq approximation [1-4], for which it is assumed that changes in density are so small that the condition $\delta\rho/\rho \leq 1$ is satisfied, i.e., the medium is, in essence, assumed to be incompressible. This is a good approximation for liquids, but for gases it imposes a substantial constraint on the temperature drop, which should also satisfy the relation $\delta T/T \ll 1$. With an arbitrary temperature drop, the changes in density are not small, and in this sense a gaseous medium cannot be considered incompressible. At the same time, solution of a complete system of gas dynamic equations that describes among other things the propagation of sound waves is impossible, since typical velocities in convection are by orders of magnitude lower than the sound velocity and the time step of numerical schemes is limited by the most rapid process allowed for by the scheme.

Simple estimations show that for low velocities of motion, pressure in the system is almost constant over the space and it can be represented as $p(x, t) = p_0(t) + \hat{p}(x, t)$, where $p_0(t)$ is a constant-over-the-space value that is dependent on time, owing to the total heating or cooling of the system, and $\hat{p}(x, t)$ is small. This small correction to pressure can be allowed for only in the equation of motion, and the local density of the gas is determined by $p_0(t)$, and the local temperature, by the equation of state. The condition of consistency of the equation of motion and the continuity equation permits the formulation of an equation for $\hat{p}(x, t)$. The elliptical character of this equation demonstrates that in a gaseous medium described in this manner there are no sound waves, which eliminates the above difficulties of choosing the time step. Essentially, the method proposed is a generalization to gaseous media of the known Harlow method [5], which is widely used to calculate incompressible fluid motion.

1. Equations of Convection in a Gaseous Medium. Convection in a gas is described by a gas dynamic equation with viscosity and heat conduction allowed for [1]:

$$\frac{D\rho}{Dt} = -\rho \operatorname{div}(\mathrm{v}), \tag{1}$$

$$\rho \frac{D\mathrm{v}}{Dt} = \rho \mathrm{g} - \nabla p + \mathrm{Q}, \tag{2}$$

$$\rho c_p \frac{DT}{Dt} = \beta T \frac{Dp}{Dt} + \operatorname{div}(k\nabla T) + \Phi, \quad \beta = -(\partial\rho/\partial T)_p/\rho. \tag{3}$$

The gas pressure p is found from ρ and T using the equation of state

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$$p = p(\rho, T).\tag{4}$$

Terms associated with viscosity are determined by the following expressions [2]:

$$Q_i = \frac{\partial}{\partial x_k}\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right),\tag{5}$$

$$\Phi = \eta\left(\frac{\partial v_l}{\partial x_k}+\frac{\partial v_k}{\partial x_l}-\frac{2}{3}\delta_{lk}\frac{\partial v_m}{\partial x_m}\right)\frac{\partial v_l}{\partial x_k},\tag{6}$$

Summation over recurrent indexes k, l, and m is understood. The system of equations is closed by the boundary conditions

$$v_i\big|_{\text{at the boundary}} = 0, \qquad T\big|_{\text{at the boundary}} = T(t).\tag{7}$$

To simplify the initial equations, we perform simple estimate, assuming that convection occurs in the gaseous medium. If $\Delta\rho$ is the characteristic change in density owing to heating, the heated gas, floating up at height h owing to the Archimedean force, acquires a velocity on the order of $v \sim \sqrt{gh\Delta\rho/\rho}$. Actually, due to the influence of viscosity and the inflow of a cold gas, which make the motion slower, this velocity can be noticeably lower, which justifies our estimates to an even greater extent. When the condition

$$\varepsilon = \frac{\mu g h}{RT} \ll 1\tag{8}$$

is satisfied, v turns out to be much lower than the sound velocity $c \sim \sqrt{RT/\mu}$. Under the same conditions, the hydrostatic pressure $p_h \sim \rho g h$ and the dynamic pressure $p_d \sim \rho v^2$ turn out to be much lower than the total pressure:

$$p_d \leq p_h \ll p.\tag{9}$$

Therefore, with a high degree of accuracy, the pressure is equal to

$$p(\mathbf{x}, t) = p_0(t) + \hat{p}(\mathbf{x}, t),\tag{10}$$

with $\hat{p} \ll p_0$ and p_0 as a function of time being determined by the total heating of the gas. Since condition (8) is satisfied under laboratory conditions with a good safety margin, we can simplify the initial equations, retaining the pressure correction $\hat{p}$ only in the equation of motion (2). With the same degree of accuracy, we can ignore the energy dissipation due to viscosity in the heat-conduction Eq. (3). Having performed the indicated rearrangements, we obtain

$$\frac{D\rho}{Dt} = -\rho\,\mathrm{div}(\mathbf{v}),\tag{11}$$

$$\rho\frac{D\mathbf{v}}{Dt} = (\rho-\rho_0)\,\mathbf{g} - \nabla\hat{p} + \mathbf{Q},\tag{12}$$

$$Q_i = \frac{\partial}{\partial x_k}\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}d\right),\tag{13}$$

$$\rho c_p\frac{DT}{Dt} = \beta T\frac{dp_0}{dt} + \mathrm{div}(k\nabla T),\tag{14}$$

$$p_0(t) = p(\rho, T),\tag{15}$$

where ρ_0 is gas density averaged over the volume.

The rearrangements made are of consequence. First, from Eq. (15) we see that the gas density is unambiguously determined by the temperature and the equation of state of the gas, ρ_0 being determined by the law of conservation of mass in the volume

$$M = \int \rho dV. \tag{16}$$

The pressure correction $\hat{p}$ does not enter in the equation of state and has to be determined by other considerations. Namely, by calculating the divergence from Eq. (12) we obtain an elliptical equation for $\hat{p}$. The method for solving these equations will be considered in detail later, and now we note only the most important consequence of the approximations made: sound waves do not propagate in the medium described by Eqs. (11)-(15). Since $v << c$ this does not have any physical meaning but is of great importance in the adoption of numerical methods.

For small temperature drops $T' = T - T_0 << T$, further simplifications of the system of equations are possible. In this case, $\rho - \rho_0 = -\rho_0\beta T' << \rho_0$. By discarding the change in density in heating everywhere except in the term with the lift, assuming k and y to be constant over the volume, and omitting the index in ρ_0 we obtain the system of equations

$$\nabla \mathrm{v} = 0, \tag{17}$$

$$\frac{D\mathrm{v}}{Dt} = -\nabla\frac{\hat{p}}{\rho} + \nu\Delta\mathrm{v} - \beta \mathrm{g} T', \tag{18}$$

$$\frac{DT'}{Dt} = \chi\Delta T', \tag{19}$$

where $\rho = \eta/\rho$ is the kinematic viscosity and $\chi = k/\rho c_p$ is the thermal diffusivity. The equations of convection in the form of (17)-(19) are known as the Boussinesq approximation [1-4] and are widely used to analyze convection in liquids. Qualitative analysis of Eqs. (17)-(19) shows [2, 6] that in this case the character of flow is determined by the Rayleigh number

$$\mathrm{Ra} = \frac{g\beta T' h^3}{\nu\chi}, \tag{20}$$

where h is the characteristic dimension of the volume.

The present work seeks to extend the numerical method of calculating unsteady convection in gases for an arbitrary temperature drop and to perform calculations under conditions close to those of an experiment [7] in which different types of instabilities were observed.

2. Numerical Method for Solution of Equations. The numerical methods adopted are substantially different for incompressible liquids and for gases. Although, to solve gas dynamic Eqs. (1)-(4), there is a plethora of different methods, they are the same in essence. If we know all characteristics of the flow at an instant t by calculating, using the equation of state (4), the forces that act on each element of the medium (1), we can determine the

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* In Eq. (12), we replaced

$$\hat{p} \to \hat{p} + \rho_0 \mathrm{gx} - \eta\frac{2}{3}\frac{\partial v_l}{\partial x_l},$$

where ρ_0 is the average-over-the-volume density of the gas which is, apparently, independent of time. This replacement somewhat simplifies the form of the equation. Furthermore, the very possibility of arbitrarily redetermination using this replacement of the diagonal part of the tensor of viscous stresses demonstrates that for the approximations made the factor of the *second* viscosity [2], which we did not take into account in the equations, drops out completely from the solution.

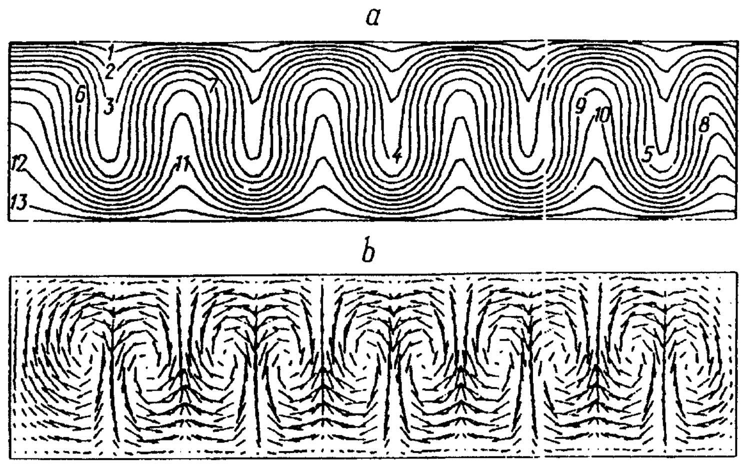

Fig. 1. (a) Stationary distribution of the temperature: 1) 301 K, 2) 323, 3) 346, 4) 368, 5) 390, 6) 413, 7) 435, 8) 457, 9) 480, 10) 502, 11) 524, 12) 546, 13) 569 K. (b) Stationary velocity field (the maximum velocity is 0.1 m/sec).

characteristics of the flow at the instant $t + \Delta t$. In all these methods, there is a limitation on the time step, which is called the Courant criterion [5]:

$$\Delta t < \frac{\Delta x}{c + v}, \tag{21}$$

where Δx is the dimension of the space cells, and c is the sound velocity. This criterion is very stringent as applied to problems of convection in gaseous media, since the characteristic velocities observed in convection are by orders of magnitude lower than the sound velocity.

To calculate liquid motion, a method [5] is known using which we can eliminate pressure from system of Eqs. (17)-(19). The essence of this method is as follows: if it is assumed that the pressure is known at all points of flow at the instant t, as is done for gaseous media, we can calculate the velocities of the liquid at the instant $t + \Delta t$ using Eq. (18). If it is required now that these velocities satisfy continuity Eq. (17), we obtain a closed elliptical-type equation for the pressure. By solving this equation we can actually find the velocities at the instant $t + \Delta t$. As it turns out [5], for this method, the sound velocity drops out of the criterion on the time step (21), which is a consequence of the absence of sound waves in the solution of Eqs. (11)-(15) and (17)-(19). In the present work, this method is generalized for calculating slow ($v << c$), axisymmetric flows of a compressible gas allowing for heat conduction, viscosity, and gravitational force described by Eqs. (11)-(15). A similar like method for plane flows is described in [8, 9].

If we know all parameters of the gas at the instant t^n, the parameters at the instant $t^{n+1} = t^n + \Delta t$ are calculated in two steps. First we calculate T^{n+1} in the entire volume using the solution to the heat-conduction equation

$$\rho^n c_p^n \left(\frac{T^{n+1} - T^n}{\Delta t} + \mathbf{v}^n \nabla T^{n+1} \right) = \beta^n T^n \left(\frac{dp_0}{dt} \right)^n + \operatorname{div} (k^n \nabla T^{n+1}). \tag{22}$$

If we confine ourselves to the gas equation

$$p = \frac{\rho R T}{\mu}, \tag{23}$$

then

$$p_0(t) = \frac{MR}{\mu} \left(\int \frac{dV}{T} \right)^{-1}. \tag{24}$$

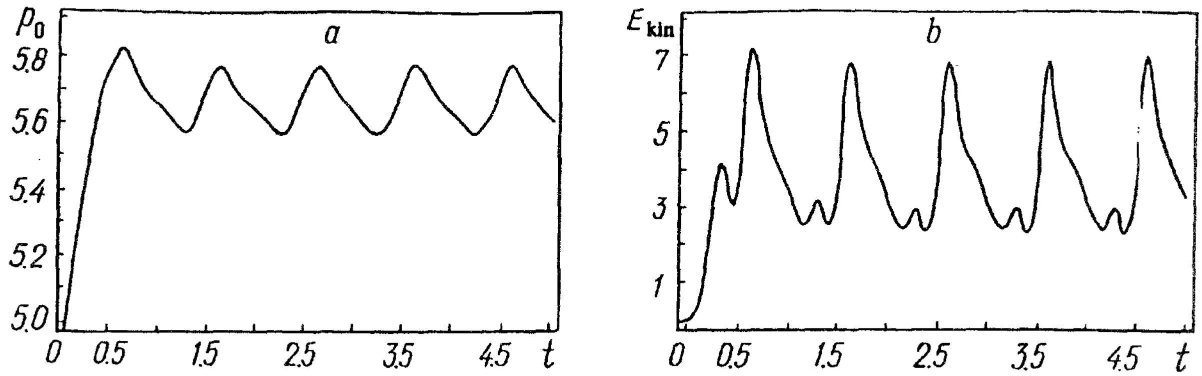

Fig. 2. (a) Pressure in a volume vs time. P_0, 10^{-1} MPa. (b) Kinetic energy in a volume vs time. E_{kin}, 10^{-8} J; t, sec.

Hence we find that

$$\frac{dp_0}{dt} = \frac{\gamma - 1}{M} \int \rho \operatorname{div}(k\nabla T)\, dV, \tag{25}$$

where $\gamma = c_p/c_v$ is the adiabatic exponent. Having determined the temperature and the density on a new time layer, we calculate the velocity. We write continuity Eq. (11) according to an implicit scheme:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = -\operatorname{div}(\rho^{n+1}\mathbf{v}^{n+1}), \tag{26}$$

and Eq. of motion (12) according to an explicit scheme:

$$\frac{\rho^{n+1}\mathbf{v}^{n+1} - \rho^n\mathbf{v}^n}{\Delta t} = (\rho^n - \rho_0)\,\mathbf{g} - \nabla\hat{p} + \mathbf{Q}^n - \mathbf{S}^n, \tag{27}$$

where $S = (\nabla v^n)\rho^n v^n$. The condition of consistency for these two equations yields an equation for the unknown $\hat{p}$:

$$\Delta\hat{p} = \frac{\rho^{n+1} - \rho^n}{\Delta t^2} + \operatorname{div}\left(\frac{\rho^n\mathbf{v}^n}{\Delta t} + \rho^n\mathbf{g} + \mathbf{Q}^n - \mathbf{S}^n\right). \tag{28}$$

If we resort once again to Eq. (27), the equation for pressure can be put into a more symmetric form:

$$\Delta\hat{p} = \frac{\rho^{n+1} - 2\rho^n + \rho^{n-1}}{\Delta t^2} + \operatorname{div}(\rho^n\mathbf{g} + \mathbf{Q}^n - \mathbf{S}^n). \tag{29}$$

By solving Eq. (28) we find, using (27), the velocities on the new time layer, which completes the calculations on one time step.

Equations (22), (27), and (28) are discretized by space variables using a grid with rectangular cells, all scalar quantities and vector divergences being determined at the *centers* of the cells and the components of the vectors and the gradients from the scalars in the middles of the corresponding *boundaries* of the cells. If a quantity is needed not at the point at which it has been determined, we perform linear interpolation. The resultant five-point equations for pressure and temperature are solved using an α–β-iteration algorithm [10].

3. Results of the Calculations. By the described algorithm we performed calculations of convection in argon that confirmed the high efficiency of the algorithm. We used the gas equation and viscosity and thermal conductivity coefficients proportional to $\sqrt{t}$. The absolute values of the coefficients were taken according to [11].

We consider convection in argon in a cylindrical plane volume with a radius of 10 cm and a height of 1 cm. The initial pressure of the argon is 0.06 MPa and the temperature is 290 K. The bottom of the volume is maintained at a temperature of 580 K, and a linear distribution of the temperature from 580 to 290 K is maintained on the volume walls. The initially quiescent gas comes into motion with heating. First, a vortex forms near the wall along

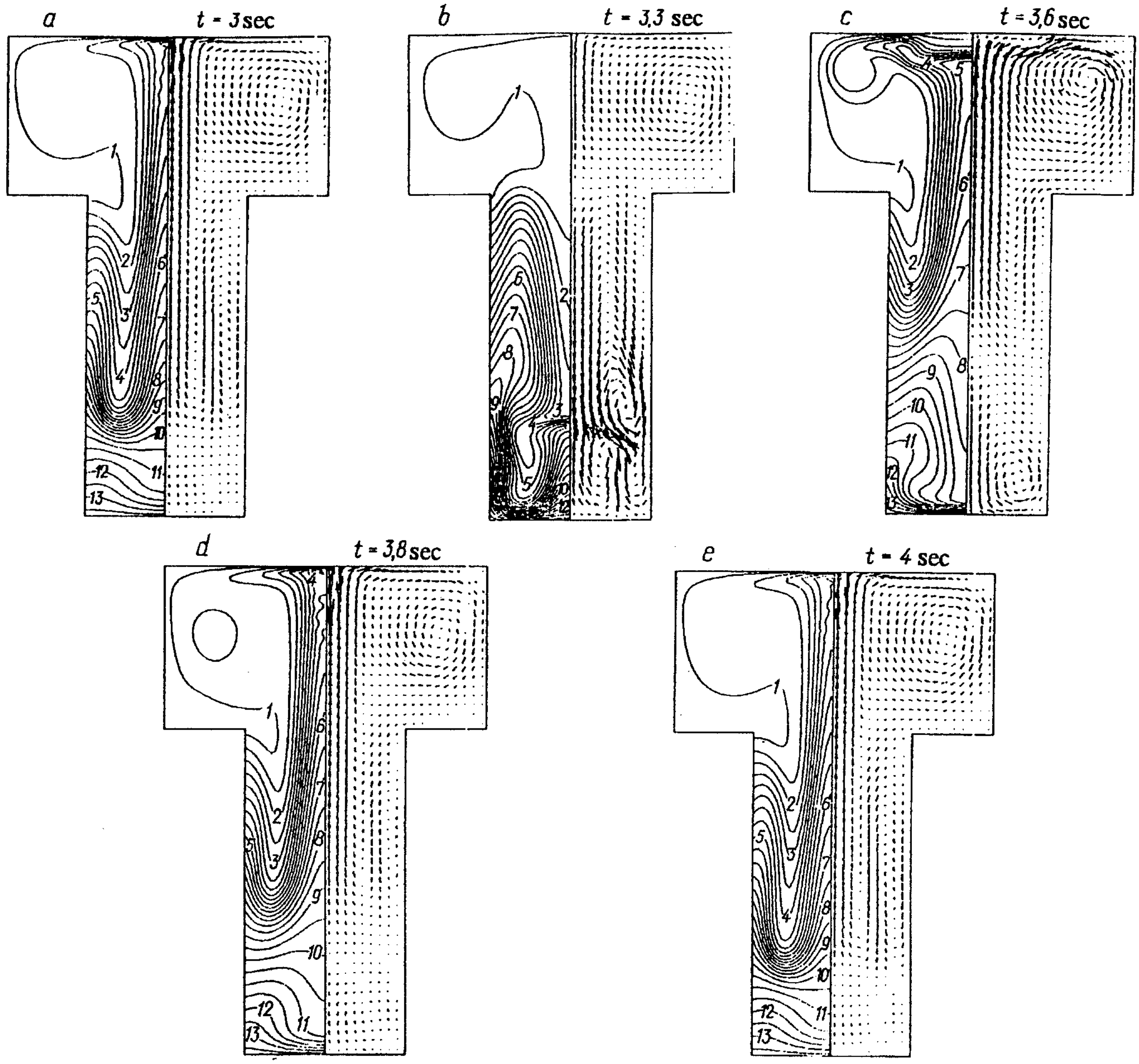

Fig. 3. Temperature and velocity fields at different time instants: 1) 301; 2) 323; 3) 346; 4) 368; 5) 390; 6) 413; 7) 435; 8) 457; 9) 480; 10) 502; 11) 524; 12) 546; 13) 569 K.

which the heated gas rises up. The initiated motion breaks the unstable equilibrium, and nine more vortices form in turn. Some time later, the motion becomes steady. The steady temperature distribution and velocity field are shown in Figs. 1a and 1b. Flow of a similar type observed experimentally in silicone oil is described in [12].

The second calculation was performed for a volume composed of two cylinders with radii of 0.5 and 1 cm and heights of 2 and 1 cm, respectively. A thin cylinder is located at the bottom. The bottom of this cylinder is maintained at a temperature of 580 K, and a linear distribution of temperature of from 580 to 290 K is maintained on its wall, while all surfaces of the wider cylinder are maintained at 290 K. The initial pressure of the argon is 0.5 MPa. Figures 2a and 2b show the pressure $p_0(t)$ and the total kinetic energy of the gas as functions of time. The character of the dependences shows clearly that under the conditions in question periodic motion of the gas with a period close to 1 sec is established. Figure 3 gives the temperature and velocity fields at different instants during one period. It is evident from the figure that the character of the flow is governed by the periodic penetration of the cold gas into the region of the operating cell, its heating, and subsequent discharge into the cold region. Comparison of Figs. 3a and 3e shows that the flows are practically identical at these instants.

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NOTATION

ρ, density; v, velocity; p, pressure; g, free fall acceleration; t, time; x, coordinate; c_p, heat capacity; T, temperature; β, thermal expansion coefficient; k, thermal conductivity coefficient; μ, atomic weight; R, universal gas constant; η, viscosity coefficient; c, sound velocity; ν, coefficient of kinematic viscosity; χ, thermal diffusivity coefficient; Ra, Rayleigh number.

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